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HW9 , Math 531, Spring 2014

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QUESTION 1. Let R be a commutative ring with $1 \neq 0$, I be a proper ideal of R, and S a multiplicatively closed set of R such that $S \cap I = \emptyset$ and $1 \in S$. Clearly $S + I$ is a multiplicatively closed set of R/I.

(i) Prove that $(R/I)_{S+I}$ is ring-isomorphic to R_S/I_S. [Hint: Define $f : (R/I)_{S+I} \to R_S/I_S$ such that $f(\frac{a+I}{s+I}) = a/s + I_S$, f is well-defined, so you don't need to check that. Now show that f is a ring-isomorphism.]

(ii) Suppose that I is a prime ideal of R. Prove that I_S is a prime ideal of R. [Hint: It is clear that a Localizing of an integral domain at a multiplicatively closed set is s an integral domain, then see (i).].

(iii) Suppose that F is a prime ideal of R_S. Prove that $F = Q_S$ for some prime ideal Q of R. [Hint : Let $Q = \{a \in R | a/s \in F$ for some $s \in S\}$, note that if $a/s \in F$, then $s(a/s) = a \in F$ and thus if $a/s \in F$, then $a/d \in F$ for every $d \in S$.]

(iv) Suppose that I is a maximal ideal of R. Prove that I_S is a maximal ideal of R_S [Hint: see (i)].

(v) Suppose that M is a maximal ideal of R_S. Prove that M need not be equal to Q_S for some maximal ideal Q of R. [Hint: Let $R = Z[X]$, $I = XZ[X]$, we know that I is a prime ideal of R that is not maximal. Let $S = R \setminus I$. Then I_S is a maximal ideal of R_S bla bla bla]

QUESTION 2. (i) Let R be a commutative ring with $1 \neq 0$ such that M_1, M_2 are the only maximal ideal of R and $M_1M_2 = \{0\}$. By a previous HW question, we know $M_1 = eR$ and $M_2 = fR$ for some idempotent elements $e, f \in R$. Let $S = \{1, e\}$. Then clearly that S is a multiplicatively closed set of R. Prove that R_S is a field.

(ii) Find an example of a ring R that satisfies the properties in (i), but R_S is ring-isomorphic to Z_2.

QUESTION 3. (i) Prove that every prime ideal of a PID is maximal.

(ii) Let F be a field such that $char(F) = p$ a prime integer. Let $a, c \in F^*$. Prove that $f(x) = x^p - ax - c \in F[X]$ has no multiple roots.

(iii) Let F be a field. Prove that $F[X]$ is a PID. Let M be a maximal ideal (prime ideal) of $F[X]$. Prove that $M = k(x)F[X]$ for some irreducible (prime) element $k(x) \in F[X]$. [Hint: recall define $L : F[X] \to N$ such that $L(f(x)) = deg(f)$, then $F[X]$ under L is a Euclidean domain....]

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